WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

A THEORY OF CONTINUOUS TRADING WHEN LUMPINESS OF CONSUMPTION IS ALLOWED†

by

Chi-fu Huang

January 1984
Revised: May 1984

WP #1574-84

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

A THEORY OF CONTINUOUS TRADING

WHEN LUMPINESS OF CONSUMPTION IS ALLOWED[†]

by

Chi-fu Huang

January 1984

Revised: May 1984

WP #1574-84

[†]Conversations with Darrell Duffie, Michael Harrison, and David Kreps proved to be very helpful. Any errors are of course my own.

Abstract

A theory of continuous trading with a very general commodity space is developed encompassing all the existing models as special cases. Agents are allowed to consume at lumps if they choose to. The martingale characterization of an equilibrium price system originated by Harrison and Kreps [8] is extended to our economy. The relationship between the sample paths properties of a price system and the way information is revealed as studied by Huang [10] is examined. In particular, we show that if agents' preferences are continuous enough, if the information of the economy is generated by a Brownian motion, and if the accumulated dividends process of a claim is continuous, then the equilibrium price process for this claim in units of the consumption commodity is an Ito process.

Abstract

A theory of continuous trading with a very general commodity space is developed encompassing all the existing models as special cases. Agents are allowed to consume at jumps if they choose so. The martingale characterization of an equilibrium price system established by Harrison and Kreps [9] is extended to our economy. The relationship between the sample paths properties of a price system and the way information is revealed as studied by Huang [10] is examined. In particular, we show that if agents' endowments are sufficiently smooth, if the information in the economy is generated by a Brownian motion, and if the accumulated dividends process of a stock is continuous, then the equilibrium price process for this stock in units of the consumption commodity is at all times continuous.

1. Introduction and Summary

The purpose of this paper is threefold. First, a theory of continuous trading in an economy with a time span [0,T] is developed. Agents in this economy are allowed to consume at any time $t \in [0,T]$. The objects of choice are processes of bounded variation representing an agent's accumulated net trades. The traditional way of modeling consumption over time, namely, the objects of choice being consumption rates, is just a special case of our economy. If an agent chooses, he can in fact consume at lumps. The issues here are what types of trading strategies are admissible and what is the appropriate formulation of the budget constraint.

Second, assuming that there exists an equilibrium in our continuous trading economy, we characterize the equilibrium price system for traded consumption claims by demonstrating its connection to martingales. A relationship similar to the one originated by Harrison and Kreps [8] is derived.

Third, the relationship between the equilibrium price system for traded claims and the way information is revealed as studied by Huang [10] is extended to the context of our economy. Keeping the setup of the economy as general as possible, we are able to characterize the sample path properties of equilibrium price processes relative to that of a traded consumption claim, the numeraire security. Roughly stated, if information is revealed continuously, and if the accumulated dividends process of a claim is contiuous, then the equilibrium price process of this claim, relative to that of the numeraire security, is continuous. The behavior of the "relative" price system when information is not revealed in a continuous manner or when the accumulated dividends process is not continuous can also be derived. Since this exercise is a straightforward application of the results in Section 6 of Huang [10], we leave it to interested readers.

In an economy where the consumption good is not available all the time as in Harrison and Kreps [8] or Huang [10], it is not unnatural to choose a traded asset to be the numeraire. Since the single consumption good is available for consumption all the time in our economy, it is therefore important to have characterizations of the equilibrium price system in units of this natural numeraire. Would the equilibrium price process of a claim having a continuous accumulated dividends process be continuous <u>in units of the consumption good</u> when information is continuous? The answer is yes if agents' preferences are continuous enough, the meaning of "enough" to be made precise. In that case, the price process is an Ito process when the information is a Brownian filtration. Agents here are allowed to consume at lumps, but as long as their preferences are very "continuous", the Ito processes representation of equilibrium price processes is an inherent property of the Brownian motion information.

Section 2 of this paper formulates a continuous time frictionless pure exchange economy under uncertainty with a time span $[0,T]$. Agents in this economy are endowed with a common information structure, a filtration, satisfying the <u>usual conditions</u> (to be defined), denoted by $\mathbf{F}$. It is assumed that there is one perishable consumption commodity in the economy which can be consumed at any time in $[0,T]$. There are a finite number of agents in the economy. Each agent is characterized by a preference relation $\succsim_i$ on V, the space of integrable bounded variation processes adapted to $\mathbf{F}$. We interpret V to be the space of net trades. There are a finite number of consumption claims traded in the economy indexed by $n = 0,1,\ldots N$. We assume that agents' preferences are continuous in a topology τ (to be defined). Each traded claim is represented by an element of V that is non-negative and increasing, describing accumulated dividends in units of the consumption

commodity, denoted by D_n, $n = 0,1,\ldots,N$. We assume that the 0th claim does not pay dividends until T, at which time, it pays a dividend that is bounded above and below away from zero.

An agent's problem in the economy is to manage a portfolio of traded claims and a consumption plan such that the budget constraint is satisfied and that his preferences are maximized. The equilibrium concept used is Radner's [19] Equilibrium of Plans, Prices, and Price Expectations. The existence of an equilibrium in our economy is not an issue to be addressed. We assume that an equilibrium exists where the consumption good is taken to be the numeraire. The equilibrium price system for traded claims is denoted by $S = \{S_n(t); t\epsilon[0,T], n = 0,1,\ldots,N\}$.

Section 3 shows that there is a natural mapping between our dynamic economy and a static Arrow-Debreu type economy but not necessarily of complete markets. It is shown that the equilibrium price system for traded claims can be represented by a conditional expectation of a stochastic integral (Proposition 3.2). This representation turns out to be very useful for the results to follow. By way of this representation, we are able to show that the equilibrium price process for the 0th claim is bounded above and below away from zero.

We show in Section 4 that if we take the 0th claim to be the numeraire and define $S^* = S/(S_o + D_o)$, then S^* is an equilibrium price system for traded claims when the price process for the consumption good is $\alpha^* = 1/(S_o + D_o)$. If we denote the (N+1)-vector of accumulated dividends process for traded claims in units of the "new" numeraire by D^*, then Theorem 4.1 shows that the process $S^* + D^*$ is an (N+1)-vector martingale under a probability measure Q that is uniformly absolutely continuous with respect to P, which is the common probability beliefs held by agents.

Given the martingale characterization of $S^* + D^*$, it is then straightforward to apply many results in Huang [10] to characterize the sample path properties of S^* in relation to different information structures. Unfortunately, the τ-continuity of agents' preferences is not strong enough to allow us to say something interesting about the sample path properties of S. Recall that S is an equilibrium price system using the consumption good as the numeraire. Since S is in units of the "natural" numeraire, it is thus interesting and important to see what conditions are needed such that we can answer yes to the following question: Is the equilibrium price process for, say, the nth claim, a continuous process when information is revealed cntinuously and when D_n is a continuous process?

Section 5 fixes a continuous information structure and defines a new topology on V denoted by τ^*. An agent's preferences are τ^*-continuous if consumptions at adjacent dates in the same state of the nature are almost perfect substitutes. In that case, Corollary 5.1 renders an affirmative to the question posed in the last paragraph. In particular, when information is a Brownian filtration, any continuous price process is an Ito process (Theorem 5.1). Section 6 discusses related works. Arguments in proving the existence of an equilibrium for an autarchy economy are outlined. Concluding remarks are in Section 7.

2. The Formulation

In this section we model a continous time frictionless pure exchange economy under uncertainty with a time span $[0,T]$, where T is a strictly positive real number.

Taken as primitive in this economy is a complete probability space $(\Omega, \mathcal{F}, P)$, where each $w \in \Omega$ represents a state of nature, which is a complete specification of the history of the exogenous environment from the

beginning of time (t=0) to the end (t=T), $\mathcal{F}$ is the tribe of distinguishable events, and P is the common probability measure on the measureable space $(\Omega, \mathcal{F})$ held by agents.

Agents in the economy are endowed with a common information structure $\mathbf{F} = \{\mathcal{F}_t,\ t\varepsilon[0,T]\}$, an increasing family of subtribes of $\mathcal{F}$, that is, $\mathcal{F}_t \subset \mathcal{F}_s$ for all $t \leq S$. We assume that the filtration $\mathbf{F}$ satisfies the usual conditions:

(i) $\mathcal{F}_t = \bigwedge_{s>t} \mathcal{F}_s \quad \forall\ t\varepsilon[0,T)$; and

(ii) $\mathcal{F}_0$ contains all the P-neglibible sets;

referred to as right contiuous and complete, respectively. We will also impose the condition that agents in the end learn the true state of nature and that now is certain. Mathematically, this means that $\mathcal{F}_T = \mathcal{F}$ and that $\mathcal{F}_0$ is almost trivial.

A process Z is a function $Z\colon \Omega x[0,T] \to R$ where $Z(\cdot,t)$ is $\mathcal{F}$-measurable for each $t\varepsilon[0,T]$. A process Z is measureable if Z is $\mathcal{F}xB([0,T])$-measurable, where $B([0,T])$ denotes the Borel tribe on $[0,T]$. The process Z is denoted by $\{Z(t)\}$. The random variable $Z(\cdot,t)$ is also denoted by $Z(t)$. The process is said to be adapted if $Z(t)$ if $\mathcal{F}_t$-measurable for each $t\varepsilon[0,T]$. The process is said to be right-continuous with left limits if $t \to Z(\omega,t)$ is right continous and $\lim_{s\uparrow t} Z(\omega,s)$ exists for each $t\varepsilon(0,T]$ for each $\omega\varepsilon\Omega$.

Two processes Z_1 and Z_2 are said to be versions of each other if $P\{Z_1(t) = Z_2(t)\}=1$ for each $t\varepsilon[0,T]$. We say that Z_1 and Z_2 are indistinguishable if $P\{Z_1(t) = Z_2(t)\ \forall\ t\varepsilon[0,T]\}=1$. Two processes which are versions of each other may not be indistinguishable. However, if Z_1 and Z_2 are versions of each other and are right (or left) continuous, then they are indistinguishable. For all practical purposes, indistinguishable processes should be regarded as the same.

From now on, the term process means a measurable process adpated to $\mathbf{F}$, unless otherwise stated.

It is assumed that there is only one perishable consumption commodity in the economy which can be consumed at <u>any</u> time $t\epsilon[0,T]$. We thus take the commodity space to be the space of integrable variation processes that have right continuous paths, denoted by V. By definition, for each $v\epsilon V$ we have:

(i) $E[\int_{[0,T]} |dv(t)|] < \infty$,

where the integral takes into account the jump of v at zero;

(ii) $v(t)$ is $\mathcal{F}_t$-measurable for all $t\epsilon[0,T]$; and

(iii) for every $\omega\epsilon\Omega$, $v(\omega,t)$ is a right coninuous function of t.

The interpretation is that $v(\omega,t)$ denotes the accumulated net trades from time 0 to time t in state $\omega\epsilon\Omega$. By convention, each $v\epsilon V$ denotes a family of indistinguishable processes. Let V_+ denote the space of nonnegative integrable increasing processes with right continuous paths. It is clear that $V_+ \subset V$. If we order the vector space V by $v_1 \geq v_2$ if and only if $v_1 - v_2 \epsilon V_+$, V_+ is the positive cone of the ordered vector space V.

<u>REMARK 2.1</u>: Since for each $v\epsilon V$, $v(\omega,t)$ is right continous and of bounded variation with respect to t for all $\omega\epsilon\Omega$, each $v\epsilon V$ is an RCLL (for right continous with finite left limits) process.

<u>REMARK 2.2</u>: It is easily checked that if $v\epsilon V$ then $E[v(t)] < \infty$ by noting that $E(\int_{[0,T]}|dv(t)|) = E(\int_{[0,T]}dv^+(t)) + E(\int_{[0,T]}dv^-(t)) = E(v^+(T)) + E(v^-(T)) < \infty$, where v^+, $v^- \epsilon V_+$ and $v^+ - v^- = v$, and that $E(v(T)) = E(v^+(T)) - E(v^-(T))$.

Before proceeding, some definitions are in order. Let X be the space of RCLL processes adapted to **F** with the property that for each $x \in X$

$$\operatorname*{ess\,sup}_{w,t} |x(\omega,t)| < \infty \tag{2.1}$$

Let X_+ be the space of RCLL processes satisfing (2.1) and with the property that $X(\omega,t) \geq 0$ for all $t \in [0,T]$ and almost every $\omega \in \Omega$. Letting $x \in X$, $x \geq 0$ means $X \in X_+$. Similarly x_1, $x_2 \in X$, $x_1 \geq x_2$ means $x_1 - x_2 \in X_+$. Denoting by X_{++} the space of processes such that for every $x \in X_{++}$, we have $x \in X_+$ and $P\{x \geq c\} = 1$ for some $c \in R_+$, $c \neq 0$.

Now consider the bilinear form ψ: $X x V \rightarrow R$:

$$\psi(x,v) = E(\int_{[0,T]} x(t)dv(t)).$$

Proposition A.1 in Appendix I shows that ψ separates points and places X and V in duality. Let τ be the strongest topology on V such that its topological dual is X, the Mackey topology (cf. Schaefer [20], p. 131).

Coming back to economics, we shall assume that there are a finite number of agents in the economy indexed by i= 1,2,...I. Each agent is characterized by his consumption preferences on net trades. Formally, each agent i is represented by a complete and transitive binary relation $\succsim_i$ on the net trade space V. We shall assume that $\succsim_i$ is convex, τ continuous, and strictly monotone, that is,

(i) the sets

$$\{v \in V: \ v \succsim_i \hat{v}\} \qquad \forall\ \hat{v} \in V$$

are convex;

(ii) the sets

$$\{v \in V: \ v \succsim_i \hat{v}\} \quad \text{and} \quad \{v \in V: \ \hat{v} \succsim_i v\} \qquad \forall\ \hat{v} \in V$$

are τ-closed; and

(iii) $v + y >_i v \quad \forall\, y \epsilon V_+,\ y \neq 0$, where $>_i$ is the strict preference relation derived from $\succsim_i$.

An example of a convex, τ-continuous, and strictly increasing preference is given by the utility function $U\colon V \rightarrow R$, $U(v) = E(\int_{[0,T]} x(t)\, dv(t))$, where $x \epsilon X_+$ and $P\{x>0\} = 1$.

We shall assume that there exists one agent, say, agent 1, whose preferences on net trade space can be represented by a concave functional $U_1 : V \rightarrow R$ whose gradients exist at all $v \epsilon V$. Furthermore, there is $\gamma \epsilon X$ with the property that

$$\operatorname*{ess\,inf}_{\omega,t} \quad \gamma(\omega,t) > 0$$

such that $\gamma \leq \nabla U_1(v)$ for all $v \epsilon V$, where ∇U_1 (v) denotes the gradient of U_1, at v. In words, agent 1's "marginal utility" is bounded below away from zero.

It is assumed that there are a finite number of contingent claims in zero net supply traded, indexed by $n=0,1,2,\ldots,N$. Each claim n is represented by an element $D_n \epsilon V_+$, where $D_n(\omega,t)$ denotes the accumulated dividends, in units of the consumption good, that a holder of a unit claim n from time zero to time t has received during that period in state $\omega \epsilon \Omega$. We shall assume that $D_0(t) = 0\ \forall\ t \epsilon [0,T)$ and $D_0(T)$ is bounded above and below away from zero. All the contingent claims are traded ex-dividends. Thus, without loss of generality, we assume that $D_n(0) = 0\ \forall\ n=1,2,\ldots,N$. Let V_0 denote the subspace of V such that $v \epsilon V_0$ implies $v(0) = 0$. Then $D_n \epsilon V_0 \cap V_+\ \forall\ n$.

An admissible price system for the single consumption good is a strictly positive bounded process $\alpha = \{\alpha(t)\}$ having RCLL paths. A consumption plan is an element of V.

An admissible price system for traded claims is an $(N+1)$-vector of integrable semimartingales[1/] adapted to $\mathbf{F}$ denoted by $S = \{S_n(t);\ n=0,1,\ldots,N\}$.

Since these claims are traded ex-dividends, $S_n(T) = 0$ a.s. $\forall$ n.

A trading strategy for claims is an (N+1)-vector of predictable and locally bounded processes $\theta = \{\theta_n(t); n=0,1,\ldots,N\}$.[2/] Given admissible price systems for the consumption good and traded claims (α,S), a trading strategy θ is said to be admissible if it satisfies the following conditions:

(i) $$\operatorname*{ess\,sup}_{n,\omega,t} |\theta_n(\omega,t)| < \infty ; \tag{2.1}$$

(ii) there exists $C\epsilon V_0$ such that if we put

$$\hat{D}(t) = \int_0^t \alpha(s)dD(s) \quad \forall\ t\epsilon[0,T] \tag{2.2}$$

and

$$\hat{C}(t) = \int_0^t \alpha(s)dC(s) \quad \forall\ t\epsilon[0,T] \quad , \tag{2.3}$$

we have

$$\theta(t)\cdot S(t) =$$

$$\theta(0)\cdot S(0) + \int_0^t \theta(s)\cdot dS(s) + \int_0^{t-} \theta(s)\cdot d\hat{D}(s) - \hat{C}(t-) \ \forall\ t\epsilon[0,T] \text{ a.s.} \tag{2.4}$$

and

$$\theta(T)\cdot\Delta\hat{D}(T) = \Delta\hat{C}(T) \quad \text{a.s.} \tag{2.5}$$

where "$\cdot$" denotes inner product, $D = \{D_n(t); n=0,1,\ldots,N\}$, and $\Delta\hat{D}(t)$ and $\Delta\hat{C}(t)$ denote the jumps of $\hat{D}$ and $\hat{C}$ at t, respectively.

Since θ is predictable, we can think of $\theta(t)$ as the portfolio held from t- to t before trading at t takes place. Thus $\theta(t)\cdot S(t)$ is the value at time t of $\theta(t)$, excluding dividends received at t and before trading and consumption take place. The right-hand-side of (2.4) is the initial value of

the strategy θ plus accumulated capital gains up to and including time t, and dividends received up to but not including time t, and minus the accumulated withdrawal up to but not including time t. The equality between $\theta(t)\cdot S(t)$ and the right-hand-side of (2.4) is just the natural budget constraint. Eq. (2.5) is the budget constraint at T. Here we only hope that the interpretaton above makes intuitive sense. It will be shown later that Eqs. (2.4) and (2.5) are just the "right" formulation.

Let $\Theta[\alpha,S]$ denote the space of admissible trading strategies with respect to the pair of admissible price systems (α,S). It is easily verified that $\Theta[\alpha,S]$ is a linear space by the linearity of the stochastic integral. By the definition, for each $\theta\varepsilon\Theta[\alpha,S]$ there is $C\varepsilon V_0$ such that (2.4) and (2.5) are satisfied, which will be called the net trade process or the consumption claim _generated_ by θ.

REMARK 2.3: Since $\theta\varepsilon\Theta[\alpha,S]$ is predictable and locally bounded, the stochastic integral in (2.4) is well-defined. See Meyer [18] for details.

An agent's problem in this economy is to manage a portfolio of traded claims and a consumption plan in order to maximize his preferences on net trades.

An _equilibrium of plans, prices and price expectations_ (of Radner [19]) is a pair of admissible price systems (α,S) for the consumption good and traded claims, and I pairs of admissible trading strategies and consumption plans, $(\theta_i,C_i)_{i=1}^{I}$, one for each agent such that C_i is $\succsim_i$-maximal in the set $\{C\varepsilon V\colon C - C(0)$ is generated by some $\theta\varepsilon\theta[\alpha,S]$, and $C(0) = -\theta(0)\cdot S(0)\}$, and markets clear; that is,

$$\sum_{i=1}^{I} \theta_i(t) = 0 \quad \forall\, t\varepsilon[0,T] \quad \text{a.s.}$$

REMARK 2.4: By the Walras Law, if the contingent claims markets clear all the time with probability one, then the market for the consumption good will clear all the time with probability one, too.

The existence of an equilibrium in the economy formulated above is not an issue to be addressed in this paper. The main purpose opf this paper is to characterize properties of an equilibrium price system when indeed one exists. Arguments for proving the existence of an equilibrium for an autarchy case are outlined, however, in Section 6. Comments are also made there on proving existence in general.

3. Properties of an Equilibirum Where the Single Consumption Good is the Numeraire

In this section we assume that there exsits an equilibrium in the economy formulated in the previous section $((\alpha,S),(\theta_i,C_i)_{i=1}^{I})$, where $\alpha(t) = 1 \; \forall \; t\varepsilon[0,T]$ a.s. That is, there exists an equilibrium using the single consumption good as the numeraire. (In fact, we will show in the sequel that there exsits an equilibrium in the economy if and only if there exists an equilibrium where the single consumption good is the numeraire.) It will be shown that this dynamic economic equilibrium can be mapped into an Arrow-Debreu type static economic equilibrium but not necessarily of complete markets. The prices in these economies are naturally linked. Their relationship will be formally characterized.

A consumption claim $v\varepsilon V_0$ is said to be marketed at time zero if v is generated by some $\theta\varepsilon\Theta[\alpha,S]$. In this case we say θ generates v. The interpretation is that if one pays $\theta(0)\cdot S(0)$ at time zero and gets into the strategy θ, one can replicate the claim v over time. When $v\varepsilon V_0 \cap V_+$, one follows θ, receives dividends, and makes withdrawals of funds out of the

portfolio according to v without making any new investment into the portfolio. When v does not belong to $V_0 \cap V_+$, additional funds are, however, needed. Let M_0 denote the space of marketed claims at time zero. By the fact that $\theta[\alpha,S]$ is a linear space, it is clear that M_0 is a linear subspace of V_0.

A current price system is a linear functional π_0: $M_0 \to R$. Let $v \epsilon M_0$ and let θ be the strategy that generates v. Define $\pi_0(v) = \theta(0) \cdot S(0)$. Then π_0 gives the price at time zero of marketed claims. Let M be the linear subspace of V such that $M \cap V_0 = M_0$. It should be clear that π_0 has an extension to M denoted by π: $M \to R$. For each $v \epsilon M$ we have $\pi(v) = v(0) + \pi_0(v - v(0))$. Then π gives the cost at time zero for all attainable net trade patterns.

A τ-continuous linear functional ψ: $V \to R$ is said to be strictly positive if $\psi(v) > 0 \ \forall \ v \epsilon V_+$ and $v \neq 0$. Each τ-continuous strictly positive linear functional can be represented by an element of X_+ that is strictly positive with probability one.

The following proposition shows that π has an extension to all of V which is τ-continous and strictly positive. Furthermore, this extension can be represented by an element of X_{++}.

PROPOSITION 3.1: π: $M \to R$ has an extension to all of V denoted by ψ, wich is τ continuous, strictly positive, and can be represented as:

$$\psi(v) = E\left(\int_{[0,T]} x(t)dv(t)\right) \quad , \tag{3.1}$$

where $x \epsilon X_{++}$, that is,

$$\operatorname*{ess\,inf}_{\omega,t} x(\omega,t) > 0 \quad .$$

PROOF: The fact that π has an extension ψ to all of V that is τ-continuous and strictly positive follows from Theorem 1 of Kreps [15]. Then ψ can be represented by an $x \epsilon X_+$ such that $P\{X>0\} = 1$. The fact that the

extension can be chosen such that ess inf $x(\omega,t) > 0$ follows from the fact that agent 1's marginal utility is bounded below away from zero. Q.E.D.

Since ψ is an extension of π, the current price of a marketed claim $v \in M_0$ is given by

$$\Psi(v) = E\left(\int_0^T x(t)dv(t)\right).$$

In particular, for traded claims, we have

$$S_n(0) = E\left(\int_0^T x(t)dD_n(t)\right) \quad n=0,1,2,\ldots,N.$$

It can be checked that each agent i's optimal net trade C_i in our dynamic economy is also $\succsim_i$-maximal in the set

$$\{C \in M: \psi(C) = \pi(C) \leq 0\}.$$

In this sense, our dynamic economic is equivalent to an Arrow-Debreu type static economy with a net trade space M and prices given by π.

The linear functional of (3.1) not only gives prices at time zero for marketed claims, but also provides a way to represent equilibrium price processes for marketed claims in our dynamic economy over time.

<u>PROPOSITION 3.2</u> Let v be a marketed claim and let $S_v(t)$ be its equilibrum price at time t. Then

$$S_v(t) = \frac{E\left(\int_t^T x(s)dv(s) \mid \mathcal{F}_t\right)}{x(t)}$$

$$= \frac{E\left(\int_0^T x(s)dv(s) \mid \mathcal{F}_t\right) - \int_0^t x(s)dv(s)}{x(t)} \quad \forall\, t \in [0,T] \text{ a.s.}, \tag{3.2}$$

where an RCLL version of the conditional expectation is taken.

PROOF: For expository purposes, we shall prove the above assertion for traded claims.

We first show that the right-hand-side of (3.2) is right continuous. This follows from the fact that we have taken a right continuous version of $E(\int_0^T x(s)dv(s) \mid \mathcal{F}_t)$, which is possible by the fact that $\mathbf{F}$ is right continuous (cf. Meyer [18], V1.T4), and that $\int_0^t x(s)dv(s)$ is right continuous.

Now consider claim n. At time zero, (3.2) is certainly true, since $x(0) = 1$ by the fact that the consumption good is taken to be the numeraire. Thus if (3.2) is not true, there must exist $t\varepsilon[0,T]$ and $A\varepsilon\mathcal{F}_t$ with $P(A) > 0$ such that (3.2) is not true for t on A, since the left-hand-side and the right-hand-side of (3.2) are both right continuous. Without loss of generality, we assume that on A:

$$S_n(t) > \frac{E(\int_t^T x(s)dD_n(s) \mid \mathcal{F}_t)}{x(t)} .$$

We define

$$\theta_k(\omega,s) = 0 \quad \forall\ \omega\varepsilon\Omega,\ \ s\varepsilon[0,T],\ k=0,1,\ldots,n-1,n+1,\ldots,N,$$

$$\begin{aligned}
\theta_n(\omega,s) &= 0 && \forall\ \omega\varepsilon\Omega,\ s\varepsilon[0,t] \\
&= -1 && \forall\ \omega\varepsilon A,\ s\varepsilon(t,T] \\
&= 0 && \forall\ \omega\varepsilon\Omega\backslash A,\ s\varepsilon(t,T] \\
C(\omega,s) &= 0 && \forall\ \omega\varepsilon\Omega,\ s\varepsilon[0,t) \\
&= S_n(\omega,t) - (D_n(\omega,s) - D_n(\omega,t)) && \forall\ \omega\varepsilon A,\ s\varepsilon[t,T] \\
&= 0 && \forall\ \omega\varepsilon\Omega\backslash A,\ s\varepsilon[t,T].
\end{aligned}$$

Thus defined, θ is predictable and locally bounded since it is left continuous. C is an integrable variation process adpapted to $\mathbf{F}$ having RCLL paths. We claim that θ generates C. To see this, we first note that for $s\varepsilon[0,T]$

and on $\Omega\backslash A$, and for $s\varepsilon[0,t]$ on A we have:

$$\theta(s)\cdot S(s) = \theta_n(s)\cdot S_n(s)$$

$$= \theta_n(0)\cdot S_n(0) + \int_0^s \theta_n(u)\, dS_n(u) + \int_0^{s-} r_n(u)\, dD_n(u) - \int_0^{s-} dC(u)$$

$$= \theta(0)\cdot S(0) + \int_0^s \theta(u)\cdot dS(u) + \int_0^{s-} \theta(u)dD(u) - \int_0^{s-} dC(u)$$

$$= 0 \quad .$$

On A and for $s\varepsilon(t,T]$, we have

$$\theta(0)\cdot S(0) \;+\; \int_0^s \theta\,(u)\cdot dS(u) + \int_0^{s-} \theta(u)\cdot dD(u) - \int_0^{s-} dC(u)$$

$$= S_n(t) - S_n(s) + D_n(t) - D_n(s-) - C(s-)$$

$$= S_n(t) - S_n(s) + D_n(t) - D_n(s-) - S_n(t) + D_n(s-) - D_n(t)$$

$$= -\, S_n(s) = \theta_n(s)\cdot S_n(s) = \theta(s)\cdot S(s) \quad .$$

It is also clear that $\theta(T)\cdot\Delta D(T) = \Delta C(T)$. Thus θ generates C and C is marketed. Since $\theta(0)\cdot S(0) = 0$, the price for C at time zero is 0. But note the following:

$$\psi(C) = \;E\;\left(\int_0^T x(s)dC(s)\right)$$

$$= \;E\;\left(x(t)\cdot C(t) - \int_t^T x(s)dD_n(s)\right)$$

$$= \;E\;\left(x(t)\cdot S_n(t) - \int_t^T x(s)dD_n(s)\right)$$

$$> E\;\left(\frac{x(t)\;\;E\left(\int_t^T x(s)dD_n\,(s)\mid{}_t\right)}{x(t)} - \int_t^T x(s)dD_n(s)\right)$$

$$= E\left(\int_t^T x(s)dD_n(s)\right) - E\left(\int_t^T x(s)dD_n(s)\right) = 0 \quad ,$$

a contradiction. Therefore we must have

$$S_n(t) = \frac{E\left(\int_t^T x(s)dD_n(s) \mid \mathcal{F}_t\right)}{x(t)} \qquad \text{a.s. } \forall\, t\varepsilon[0,T] \ .$$

Since both sides of the above expressions are right continuous, we thus proved what was to be shown. Q.E.D.

Several corollaries are immediate.

<u>COROLLARY 3.1</u>: The process $\{S_0(t)+D_0(t)\}$ is bounded above and below away from zero.

<u>PROOF</u>: From Proposition 3.2 we know, $\forall\, t\varepsilon[0,T)$,

$$S_0(t) = \frac{E\left(\int_t^T x(s)dD_0(s) \mid \mathcal{F}_t\right)}{x(t)}$$

$$= \frac{E\left(x(T)D_0(T) \mid \mathcal{F}_t\right)}{x(t)} \qquad \text{a.s.} \qquad (3.3)$$

By the hypothesis we know that $D_0(T)$ is bounded above and below away from zero and that x is bounded above. Proposition 3.1 shows that x is also bounded below away from zero. Thus $S_0(t)$ is bounded above and away from zero, and so is $S_0 + D_0$. Q.E.D.

<u>COROLLARY 3.2</u>: The process $\{x(t)\}$ is a semimartingale.

<u>PROOF</u>: From (3.3) we can write:

$$x(t) = \frac{E(x(T)D_0(T)|\mathcal{F}_t)}{S_0(t) + D_0(t)} \qquad \forall\ t\varepsilon[0,T] \qquad \text{a.s.}$$

The numerator of the above expression is a martingale and the denominator is a semimartingale, which is bounded above and below away from zero by Corollary 3.1. A generalization of Ito's formula shows that $\{x(t)\}$ is a semimartingale (cf. Meyer [18], p. 301). Q.E.D.

COROLLARY 3.3: Let $v\varepsilon M_o$ and let $\{S_v(t)\}$ be its price process. Then $\{S_v(t)\}$ is a semimartingale. Moreover, let θ be the strategy that generates v. Then, $S_v(t) = \theta(t)\cdot S(t) + \theta(t)\cdot\Delta D(t) - \Delta v(t)$ $\forall\ t\varepsilon[0,T]$ a.s.

PROOF: We know from Proposition 3.2 that

$$S_v(t) = \frac{E(\int_0^T x(s)dv(s)|\mathcal{F}_t) - \int_0^t x(s)dv(s)}{x(t)} \qquad \forall\ t\varepsilon[0,T] \quad \text{a.s.}$$

The numerator is a semimartingale since it is the sum of a martingale and a process of bounded variation. Corollary 3.2 shows that the denominator is a semimartingale too. An application of the generalized Ito's formula shows that $\{S_v(t)\}$ is a semimartingale.

Finally, the fact that $S_v(t) = \theta(t)\cdot S(t) + \theta(t)\cdot\Delta D(t) - \Delta v(t)$ follows from a no arbitrage argument. Q.E.D.

We started this section by assuming that there exists an equilibrium in our dynamic economy. Then, the only information we had was that the equilibrium price system for traded claims is a vector of semimartingales. By bridging through a static economy we have been able to say a bit more.

4. Properties of Equilibrium Price Processes

In an economy where agents can only consume at two dates, and therefore traded assets do not pay "dividends" until the end, Harrison and Kreps [8] and later Huang [10] have shown that if a vector price process is a vector equilibrium price process, then, using the price process of one of the traded assets as the numeraire, the "normalized" vector price process is a vector of martingales under an equivalent probability measure. In carrying out their analyses, they assumed that there is one traded asset whose equilibrium price process is bounded above and below away from zero.

In this section, we will give a martingale characterization of equilibrium price processes in our economy that is similar to that of Harrison and Kreps [8] and Huang [10]. By using the equilibrium price process for the 0th traded claim plus its accumulated dividends as the numeraire, we show that the vector of equilibrium price processes for traded claims plus their accumulated dividends, all in units of the numeraire, is a vector of martingales under a probability measure on $(\Omega, \mathcal{F})$ that is uniformly absolutely continuous with respect to P.

Recall from the last section that $((\alpha,S), (\theta_i,C_i)_{i=1}^{I})$ is an equilibrium, where $\alpha(t) = 1 \ \forall \ t\varepsilon[0,T]$, a.s. We shall show that $((\alpha^*,S^*), (\theta_i,C_i)_{i=1}^{I})$, where

$$\alpha^*(t) = \frac{1}{S_0(t) + D_0(t)} \quad \forall \ t\varepsilon[0,T], \quad \text{a.s.}$$

and

$$S_n^*(t) = \alpha^*(t)S_n(t) \ \forall \ t\varepsilon[0,T] \quad n = 0,1\ldots,N, \quad \text{a.s.} \tag{4.1}$$

is also an equilibrium. We first show that S* is an admissable price system.

PROPOSITION 4.1: S* is an admissible price system for traded claims.

PROOF: First we note that $S_0 + D_0$ is a semimartingale since a semimartingale plus a process of bounded variation is a semimartingale. Second, since $S_0 + D_0$ is bounded below away from zero (cf. Corollary 3.1), α^* is a semimartingale. This follows from a generalization of the Ito's Lemma. (See Meyer [18], p. 301). Third, since the product of two semimartingales is a semimartingale (cf. Corollary 23 in Chapter IV of Meyer [18]), S^* is a vector of semimartingales. Finally, the integrability of S^* follows from the fact that S is an equilibrium price system and the fact that $S_0 + D_0$ is bounded below away from zero. Q.E.D.

The following proposition is a direct consequence of Corollary 3.1:

PROPOSITION 4.2: α^* is an admissible price system for the consumption good.

PROOF: This also follows from the generalized Ito's formula. Q.E.D.

Now we put

$$D^*(t) = \int_0^t \alpha^*(s)dD(s) \quad \forall\ t \in [0,T]$$

and

$$v^*(t) = \int_0^t \alpha^*(s)dv(s) \quad \forall\ t \in [0,T]\ v \in V \quad .$$

Thus, $D^*(t)$ is the vector accumulated dividends of traded claims and $v^*(t)$ is the accumulated net trade, from time zero to time t in units of $S_0 + D_0$. Let $\Theta[\alpha^*,S^*]$ denote the space of admissible trading strategies when the price systems for the consumption good and the traded claims are (α^*,S^*).

<u>PROPOSITION 4.3</u>: $\theta\varepsilon\Theta[\alpha,S]$ if and only if $\theta\varepsilon\Theta[\alpha^*,S^*]$. Furthermore, $v\varepsilon V_0$ is marketed given (α,S) if and only if v is marketed given (α^*,S^*).

<u>PROOF</u>: Let $\theta\varepsilon\Theta[\alpha,S]$. Then we know θ is predictable and bounded. In addition, there is $v\varepsilon V_0$ such that:

$$\theta(t)\cdot S(t) = \theta(0)\cdot S(0) + \int_0^t \theta(s)\cdot dS(s) + \int_0^{t-} \theta(s)\cdot dD(s)$$

$$- \int_0^{t-} dv(s) \quad \forall\, t\varepsilon[0,T) \quad \text{a.s.} \tag{4.2}$$

and

$$\theta(T)\cdot\Delta D(T) = \Delta v(T) \quad \text{a.s.} \tag{4.3}$$

We want to demonstrate that there is $v'\varepsilon V_0$ such that

$$\theta(t)\cdot S^*(t) = \theta(0)\cdot S^*(0) + \int_0^t \theta(s)\cdot dS^*(s) + \int_0^{t-} \theta(s)\alpha^*(s)\cdot dD(\tau)$$

$$- \int_0^{t-} \alpha^*(s)dv'(s) \quad \forall\, t\varepsilon[0,T] \quad \text{a.s.} \tag{4.4}$$

and

$$\theta(T)\cdot\Delta D(T) = \Delta v'(T) \quad \text{a.s.} \tag{4.5}$$

We claim that we can take v' to be v. Eq. (4.5) is obviously satisfied by taking $v' = v$. Next we define

$$G(t) = \theta(t)\cdot\Delta D(t) - \Delta v(t)$$

and

$$G^*(t) = \theta(t)\alpha^*(t)\cdot\Delta D(t) - \alpha^*(t)\Delta v(t) \quad .$$

Note that $G^*(t) = \alpha^*(t)G(t)$. The process

$$\theta(t)\cdot S(t) + G(t) = \theta(0)\cdot S(0) + \int_0^t \theta(s)dS(s) + \int_0^t \theta(s)dD(s)$$

$$- \int_0^t dv(s) \quad t\varepsilon[0,T] \quad \text{a.s.} \tag{4.6}$$

is a semimartingale, since the first integral on the right-hand side is a semimartingale (cf. Theorem IV.20 of Meyer [18]), and the second and the third integrals are processes of bounded variation. (This fact can also be seen from Corollary 3.3). Here we remark that $\theta(t)\cdot S(t)$ may not be a semimartingale, since if D or v is not continuous, then $\theta(t)\cdot S(t)$ is not right continuous, which is a defining property of a semimartingale. Now using the differentiation rule for semimartingales (cf. Corollary IV.23 of Meyer [18]) we have:

$$\begin{aligned} d(\theta(t)\cdot S^*(t) + G^*(t)) &= d(\alpha^*(t)(\theta(t)\cdot S(t) + G(t))) \\ &= \alpha^*(t-)d(\theta(t)\cdot S(t) + G(t)) + (\theta(t-)\cdot S(t-) \\ &\quad + G(t-))d\alpha^*(t) + d[\alpha^*,\theta\cdot S + G]_t , \end{aligned} \tag{4.7}$$

where $[\cdot,\cdot]$ denotes the joint variation process (cf. Meyer [18], p. 267). The second term on the right hand side of (4.7) can be rewritten as:

$$\begin{aligned} (\theta(t-)\cdot S(t-) + G(t-))d\alpha^*(t) &= (\theta(t)\cdot S(t) + G(t) - \theta(t)\Delta S(t) \\ &\quad - \theta(t)\Delta D(t) + \Delta v(t))d\alpha^*(t) \\ &= \theta(t)S(t-)d\alpha^*(t) , \end{aligned} \tag{4.8}$$

where the first equality follows from the definition of a stochastic integral and the second follows from the definition of G(t).

We can also rewrite the third term on the right-hand side of (4.7) as:

$$d[\alpha^*,\theta\cdot S + G]_t = \theta(t)\cdot d[\alpha^*,S] + \theta(t)\cdot d[\alpha^*,D]_t - d[\alpha^*,v]_t \tag{4.9}$$

by Theorem IV.18 of Meyer [18]. Applying the differentiation rule for semimartingales again we get:

$$dS^*(t) = d(\alpha^*(t)S(t))$$
$$= \alpha^*(t-)dS(t) + S(t-)d\alpha^*(t) + d[\alpha^*,S]_t \quad . \tag{4.10}$$

Finally, we substitute (4.8) and (4.9) into (4.7) and use (4.10) to get:

$$d(\theta(t)\cdot S^*(t) + G^*(t)) = \alpha^*(t-)\theta(t)\cdot dS(t) + \alpha^*(t-)\theta(t)\cdot dD(t) - \alpha^*(t-)dv(t)$$
$$+ \theta(t)\cdot S(t-)d\alpha^*(t) + \theta(t)\cdot d[\alpha^*,S]_t + \theta(t)\cdot d[\alpha^*,D]_t - d[\alpha^*,v]_t$$
$$= \theta(t)\cdot dS^*(t) + \theta(t)\alpha^*(t)\cdot dD(t) - \alpha^*(t)dv(t), \tag{4.11}$$

where we have used the fact that

$$\alpha^*(t-)dD(t) + d[\alpha^*,D]_t = \alpha^*(t)dD(t)$$

and

$$\alpha^*(t-)dv(t) + d[\alpha^*,v]_t = \alpha^*(t)dv(t)$$

(cf. Corollary IV.23 of Meyer [18]). Eq. (4.11) is just the differential form of (4.4) and we've proved our claim, that is, $\theta\varepsilon\Theta[\alpha^*,S^*]$.

Conversely, let $\theta\varepsilon\Theta[\alpha^*,S^*]$. A proof very similar to the above one shows that $\theta\varepsilon\Theta[\alpha,S]$.

Next, let $v\varepsilon V_0$ be marketed given (α,S). That is, there is $\theta\varepsilon\Theta[\alpha,S]$ such that θ generates v. From the above proof we know $\theta\varepsilon\Theta[\alpha^*,S^*]$ and it also generates v. The arguments for the converse are identical. Q.E.D.

Note that in Section 2 we only gave loose interpretation of the budget constraints (2.4) and (2.5). The above proposition shows that the budget constraints we posited are invariant under a change of unit. We feel therefore

that they are just the right formulation.

If we let M^* denote the space of attainable net trades given (α^*,S^*), then a direct consequence of Proposition 4.3 is:

Corollary 4.1: $v\in M$ if and only if $v\in M^*$.

Now we define $\pi^*: M^* \to R$ by $\pi^*(v) = \alpha^*(0)v(0) + \theta(0)\cdot S^*(0)\ \forall\ v\in M^*$, where $\theta\in\Theta[\alpha^*,S^*]$ and generates $v - v(0)$. π^* is a current price system for (α^*,S^*). It is clear that $\pi^* = \alpha^*(0)\cdot\pi$.

PROPOSITION 4.4: $((\alpha^*,S^*),(\theta_i,C_i)_{i=1}^I)$ is an equilibrium.

PROOF: Since $((\alpha,S),(\theta_i,C_i)_{i=1}^I)$ is an equilibrium, we know that for each i, C_i is $\succsim_i$-maximal in the set $\{v\in M: \pi(v) \leq 0\}$. From the fact that $M = M^*$ and $\pi^* = \alpha^*(0)\cdot\pi$, we know

$$\{v\in M: \pi(v) \leq 0\} = \{v\in M^*: \pi^*(v) \leq 0\} .$$

Thus C_i is also $\succsim_i$-maximal in the set

$$\{v\in M^*: \pi^*(v) \leq 0\} ,$$

and $((\alpha^*,S^*),(\theta_i,C_i)_{i=1}^I)$ is an equilibrium. Q.E.D.

Given what we have up to now, it is easy to see that we have in fact proved a strong statement: $((\alpha_1,S_1),(\theta_i,C_i)_{i=1}^I)$ is an equilibrium if and only if $((\alpha_2,S_2),(\theta_i,C_i)_{i=1}^I)$ is, where α_1/α_2 is a process bounded above and below away from zero. This is hardly a surprise, since after all only relative processes are determined in an economic equilibrium. The restriction put on α_1/α_2 is natural, however, in an economy with an infinite dimensional commodity space (cf. Duffie and Huang [5]).

It turns out that we have an interesting and useful characterization of the price systems (α^*, S^*).

THEOREM 4.1: There exists a probability measure Q on $(\Omega, \mathcal{F})$, which is uniformly absolutely continuous with respect to P, such that we have, for every $t \in [0,T]$,

$$S_n^*(t) + D_n^*(t) = E^*[S_n^*(s) + D_n^*(s) \mid \mathcal{F}_t] \quad \text{a.s.} \quad \forall\, s \geq t \ , \tag{4.12}$$

where as usual we have fixed a right continuous version of the conditional expectation. In words, the process $S^* + D^*$ is a vector martingale under Q.

When traded claims do not pay dividends, Theorem 4.1 is just the martingale result originated by Harrison and Kreps [8]. This should have been anticipated, since their economy is a special case of ours.

REMARK 4.1: Since P and Q are uniformly absolutely continuous with respect to each other, they have the same null sets. Thus all the almost surely statements from now on apply to both equally well. No distinction will be made unless otherwise needed.

A proposition is needed before we prove our main theorem.

PROPOSITION 4.4: There exists a strictly positive process x^* which is bounded above and below away from zero such that $\forall$ n

$$S_n^*(t) = \frac{E(\int_t^T x^*(s) dD^*(s) \mid \mathcal{F}_t)}{x^*(t)} \quad \forall\, t \in [0,T] \quad \text{a.s.} \tag{4.13}$$

PROOF: Since $\pi^* = \alpha^*(0) \cdot \pi$, π^* has an extension to all of V denoted by ψ^*. It is clear that ψ^* can be chosen to be $\alpha^*(0) \cdot \psi$. That is,

$$\psi^*(v) = \alpha^*(0)E(\int_{[0,T]} x(t)dv(t)) \quad v\epsilon V \ , \tag{4.14}$$

where $x\epsilon X_{++}$.

Now putting

$$x^*(t) = \frac{\alpha^*(0)}{\alpha^*(t)} x(t) \quad \forall\ t\epsilon[0,T] \tag{4.15}$$

x^* is strictly positive, bounded above and below away from zero.

We can rewrite (4.14) as

$$\psi^*(v) = E(\int_{[0,T]} x^*(t)dv^*(t)) \quad .$$

Applying the same arguments to prove Proposition 3.2, it is straightforward to verify (4.13). Q.E.D.

<u>COROLLARY 4.2</u>: The process x^* defined in (4.15) is a martingale under P.

<u>PROOF</u>: We note that

$$D_0^*(t) = 0 \quad \forall\ t\epsilon[0,T)$$

$$= 1 \qquad \text{a.s.} \qquad t = T$$

and

$$S_0^*(t) = 1 \quad \forall\ t\epsilon[0,T)$$

$$= 0 \qquad \text{a.s.} \qquad t = T \ ,$$

by construction. By Proposition 4.4 we then have

$$\frac{E(x^*(T) \mid \mathcal{F}_t)}{x^*(t)} = S_0(t) = 1 \qquad \forall\ t\epsilon[0,T) \quad \text{a.s.}$$

That is,

$$E(x^*(T) \mid \mathcal{F}_t) = x^*(t) \qquad \forall\ t\epsilon[0,T) \quad \text{a.s.}$$

Naturally $E(x^*(T) \mid \mathcal{F}_T) = x^*(T)$ a.s., since x^* is adapted. This simply says that x^* is a martingale. Q.E.D.

PROOF OF THE THEORM: From Corollary 4.2, x^* is a martingale. Equivalently stated, x^* is the optional projection of the (not adapted) process $y = \{y(t) = x^*(T);\ t \varepsilon [0,T]\}$ (cf. Theorem VI.43 of Dellacherie and Meyer [2]).

It then follows from Theorem VI.57 of Dellacherie and Meyer [2] that

$$
\begin{aligned}
S^*(t) &= \frac{E(\int_t^T x^*(\tau) dD_n^*(\tau) \mid \mathcal{F}_t)}{x^*(t)} \\
&= \frac{E(\int_t^T x^*(T) dD_n^*(\tau) \mid \mathcal{F}_t)}{x^*(t)} \\
&= \frac{E(x^*(T)(D_n^*(T) - D_n^*(t)) \mid \mathcal{F}_t)}{x^*(t)} \quad \forall\ t \varepsilon [0,T] \ \text{a.s.} \qquad (4.17)
\end{aligned}
$$

Next note that $E[x^*(T)] = S_0^*(0) = 1$. Thus if we put

$$Q(B) = \int_B x^*(\omega,T) p(d\omega) \ ,$$

Q is a probability measure on $(\Omega, \mathcal{F})$ which is uniformly absolutely continuous with respect to P. Therefore (4.17) can be written as

$$S_n^*(t) = E^*(D_n^*(T) - D_n^*(t) \mid \mathcal{F}_t) \quad \forall\ t \varepsilon [0,T] \ \text{a.s.} \ ,$$

where $E^*(\cdot)$ denotes the expectation under Q. (cf. Gihman and Skorhod [6], p. 149). Equivalently,

$$S_n^*(t) + D_n^*(t) = E^*[S_n^*(t) + D_n^*(t) \mid \mathcal{F}_t] \quad \forall\ t \geq t \quad \text{a.s.}$$

Q.E.D.

REMARK 4.2: Theorem 4.1 can also be proved by using an arbitrage

argument. The intent of the proof above is primarily methodological in demonstrating some techniques that may be useful in future works.

Several corollaries are immediate.

<u>COROLLARY 4.3</u>: S^* is a vector of supermartingales under Q.

<u>PROOF</u>: Consider, for all $t_1 \leq t_2$,

$$E^*[S^*(t_2) | \mathcal{F}_{t_1}] - S^*(t_1)$$

$$= E^*[S^*(t_2) | \mathcal{F}_{t_1}] - E^*[S^*(t_2) + D^*(t_2) | \mathcal{F}_{t_1}] + D^*(t_1)$$

$$= E^*[D^*(t_1) - D^*(t_2) | \mathcal{F}_{t_1}] \leq 0 \quad ,$$

where the first equality follows from Theorem 4.1, and where the inequality follows from the fact that $D_n \varepsilon V_+$ for all n and α^* is strictly positive.

Q.E.D.

<u>COROLLARY 4.4</u>: Let $v \varepsilon M_0$ and let S_v be its equilibrium price process. Then $S_v^* + v^*$ is a martingale under Q.

<u>PROOF</u>: Since $v \varepsilon M_0$, we can treat v as if it were traded and directly apply the theorem. Q.E.D.

<u>COROLLARY 4.5</u>: Let $v_1, v_2 \; \varepsilon \; M_0$ be such that $v_1^*(T) - v_1^*(t) = v_2^*(T) - v_2^*(t)$ for some $t \varepsilon [0,T]$. Then

$$S_{v_1}(t) = S_{v_2}(t) \quad \text{a.s.}$$

<u>PROOF</u>: From the Theorem we know

$$S_{v_i}^*(t) = E^*[v_i^*(T) - v_i^*(t) | \mathcal{F}_t] \quad \text{a.s.} \quad i = 1,2 \; .$$

From the hypothesis that $v_1^*(T)-v_1^*(t) = v_2^*(T)-v_2^*(t)$ we get $S_{v_1}^*(t) = S_{v_2}^*(t)$ a.s. Dividing through by $\alpha^*(t)$ as have $S_{v_1}(t) = S_{v_2}(t)$ a.s. Q.E.D.

At any time t, if the future accumulated dividends in units of $S_0 + D_0$ are equal for two marketed claims, they must have the same price.

Once we have the martingale characterization of (α^*, S^*) in the Theorem, it is easy to see that the sample path properties of S^* depend both upon D^* and upon how information is revealed. This observation also applies to any marketed claims by Corollary 4.4. The propositions to follow are stated in terms of marketed claims and are direct consequences of Theorem 4.1 and results of Huang [10].

DEFINITION: The information structure **F** is said to be continuous if all the martingales adapted to **F** are P-indistinguishable from continuous processes.

REMARK 4.3: The above definition for a continuous information structure is invariant under a substitution of an equivalent probability measure. For an extensive treatment of a continuous information structure, see Huang [10].

REMARK 4.4: We shall call a process indistinguishable from a continuous process a continuous process from now on.

PROPOSITION 4.5: Let $v \in M_0$. If **F** is continuous, then $S_v^* + v^*$ is a vector of continuous processes. Therefore, over any nontrivial subinterval of $[0,T]$, the sample paths of $S_v^* + v^*$ are either of unbounded variation or are constants.

PROOF: See Propositions 4.1, 4.2, and 4.3 of Huang [10]. Q.E.D.

COROLLARY 4.5: Suppose that $\mathbf{F}$ is continuous. Then S_v^* is a continuous process in a subinterval of $[0,T]$ if and only if $v \in M_0$ is.

PROOF: Suppose that $v \in M_0$ is a continuous process in $[t_1,t_2) \subset [0,T]$, say. Then v^* is a continuous process in $[t_1,t_2)$. Proposition 4.5 implies that S_v^* is continuous in $[t_1,t_2)$.

Conversely, let S_v^* be a continuous process in $[t_1,t_2) \subset [0,T]$. It follows from Proposition 4.5 that v^* is continuous there. Therefore, v must be a continuous process in $[t_1,t_2)$. Q.E.D.

Sample path properties of S_v^* for other information structures can also be easily derived. They are direct applications of Section 6 of Huang [10]. We leave this straightforward exercise to interested readers. Anticipating the results in that direction, however, we can say that the behavior of the equilibrium price process of a marketed claim, relative to that of the 0th claim, is entirely determined by its accumulated dividends process (in units of the consumption commodity) and the way information is revealed.

Note that up to now what we have been able to do is to characterize some sample path properties of equilibrium price processes numerated in an unnatural numeraire, a traded claim. In an economy like Harrison and Kreps [8] or Huang [10], consumption takes place at two time points, 0 and T. There is no consumption between those two time points. Hence, there does not exist a natural numeraire throughout the time span of the economy. In our economy, however, consumption can take place at anytime throughout $[0,T]$. It can then be argued that the single perishable consumption good should be the natural

numeraire. In the framework we have been operating, there is no general statement that can be made regarding the sample paths properties of S. Nonetheless, if we are willing to make assumptions on endogenous variables, a statement is possible.

PROPOSITION 4.6: Let $v \varepsilon M_0$, and suppose that F is continuous. Any two statements below imply the third one. In any subinterval $[t_1, t_2)$ or $[t,T]$ $[0,T]$,

(i) α^* is continuous;

(ii) S_v is continuous;

(iii) v is continuous.

PROOF: The assertion follows directly from (3.2). Q.E.D.

The most important result in the section is Theorem 4.1. If S is an equilibrium price system for traded claims, then there exists a probability measure Q on $(\Omega, \mathcal{F})$ which is uniformly absolutely continuous with respect to P such that the "relative" price system S* plus the "relative" payoff structure D* is a vector of martingales under Q. Various sample paths properties of the relative price system for marketed claims are then derived.

5. An Alternative Formulation

In the previous sections we considered an economy where agents' preferences on net trades are τ-continuous. Roughly, τ-continuity of preferences only requires that agents' "shadow prices" for consumption be adapted and bounded. A unit of consumption at time t in state ω to an agent may be very different from a unit of consumption an instant later in the same state. Thus, using the consumption as the natural numeraire, it is hardly a

surprise that continuity of the information structure does not necessarily imply continuity of the sample paths of the equilibrium price process for a traded claim even if whose accumulated dividends process in continuous.

It has been shown in Huang [10] that, in the context of his economy, equilibrium price processes for traded claims not only are continuous but can be represented as Ito processes when information is a Brownian filtration. Since the Ito process formulation of equilibrium price processes is so prevalent in the literature of financial economics, it is thus desirable to characterize conditions under which the equilibrium price processes in our economy are Ito processes.

In this section we will consider an economy which is identical to the one formulated in Section 2 except that agents' preferences are assumed to be "more" continuous in the sense that consumtion at adjacent dates are "almost" perfect substitutes, and that the information structure of agents are assumed to be continuous to start with. Then we show that the equilibrium price process for a traded claim, <u>using the consumption good as the numeraire</u>, is a continuous process. Furthermore, this continuous process is an Ito process if the continuous information structure is generated by a Brownian motion.

Let H denote the space of bounded continuous processes (adapted to $\mathbf{F}$). Define a bilinear form $\phi: V \times H \to R$ by:

$$\phi(v,h) = E\left(\int_{[0,T]} h(t)dv(t)\right).$$

It is easily seen that V separates points in H through ϕ. The fact that H separates points in V follows from the fact that when information is continuous the optional σ-field and the predictable σ-field coincide (cf. Lemma 2.2 of Chung and Williams [1] and Theorem 6.3 of Huang [10]), and Section VII.4 of Dellacherie and Meyer [2]. Thus, ϕ place V and H in duality. Let τ^* be the strongest topology on V such that its topological dual is H.

We assume that agents' preferences $\succsim_i$, i=1,2,...,I are τ^*-continuous, convex, and strictly increasing. Since τ^* is a weaker topology on V than τ, τ^*-continuity is a stronger requirement than τ-continuity. An example of a τ^*-continuous, convex, and strictly increasing preference is a utility function:

$$U: V \to R, \quad U(v) = E\left(\int_{[0,T]} f(t)dv(t)\right),$$

where $f \in H$ and is strictly positive.

Agent 1's preferences are assumed to be representable by a concave functional $U_1 : V \to R$ whose gradients exist at all $v \in V$. Furthermore, there is $\gamma^* \in H$ with the property that:

$$\operatorname*{ess\,inf}_{\omega,t} \gamma^*(\omega,t) > 0 \quad ,$$

such that $\gamma \leq \nabla U_1(v)$ for all $v \in V$. The following proposition can be viewed as a corollary of Proposition 3.2.

PROPOSITION 5.1: There exists an $h \in H$ with the properties that ess inf $h(\omega,t) > 0$ and that it is a semimartingale such that for all $v \in M_0$ we have

$$S_v(t) = \frac{E\left(\int_0^T h(s)dv(s) \mid \mathcal{F}_t\right) - \int_0^t h(s)dv(s)}{h(t)} \quad \forall\, t \in [0,T] \quad \text{a.s.} \tag{5.1}$$

where we still use M_0 to denote the space of marketed claims, where S_v is the equilibrium price process for v using the consumption good as the numeraire.

A corollary is immediate:

COROLLARY 5.1: Let $v \in M_0$. Then S_v is a semimartingale. Moreover, S_v is a continuous process on any subinterval of [0,T] if and only if v is.

Before proving the main result of this section we first record a technical lemma and give a definition for an Ito process.

LEMMA: The information structure generated by a (multidimensional) Brownian is continuous. Moreover, every local martingale Y adapted to the Brownian information can be represented as an Ito integral.

PROOF: See Theorem VIII.62 of Dellacherie and Meyer [2]. Q.E.D.

DEFINITION: A process is said to be an Ito process if it can be written as the sum of an Ito integral and a continuous bounded variation process (cf. Chapter 5 of Harrison [7]).

THEOREM 5.1: If **F** is generated by a (multidimensional) Brownian motion, then the equilibrium price process, in units of the consumption good, of a marketed claim having a continuous dividends process is an Ito process.

PROOF: The numerator of (5.1) is continuous since information is continuous and the accumulated dividends process is continuous. It follows then that the price process is a continuous semimartingale because $\{h(t)\}$ is a continuous process. The price process can be written as the sum of a local martingale and a process of bounded variation. By the Lemma, the local martingale can be represented as an Ito integral; therefore continuous. The bounded variation part is thus also continuous. Hence the price process is representable as the sum of an Ito integral and a continuous process of bounded variation and by definition is an Ito process. Q.E.D.

When information is generated by a Brownian motion, the continuity of

the equilibrium price process of a marketed claim makes the price process an Ito process.

In an economy where agents are allowed to consume at only two dates, 0 and T, the continuity of a price process is a direct consequence of the fact that information is revealed in a continuous manner. That is because agents' shadow prices for a unit "wealth" over time is then automatically a continuous process when information is continuous. When agents are allowed to consume at any time in [0,T], the shadow prices for consumption are determined by preferences and not by information. Thus, the way information is revealed does not dictate the behavior of equilibrium price processes. Very continuous preferences are needed.

6. Related Works

Huang [11] has shown that there exists an equilibrium in a continuous trading economy where agents are allowed to consume at only two dates and where information is generated by a vector diffusion process. The equilibrium price processes are Ito processes. Under some conditions, the vector of equilibrium price processes and the vector diffusion process generating information form a vector diffusion process. Duffie [4] has proved the existence of an equilibrium in an economy where agents are allowed to consume continuously at "rates" and with a more general information structure. Both, however, used the machinery developed in Duffie and Huang [5]. It is not hard to see that putting the results of Duffie [4] and Huang [11] together one can show that in Duffie's economy, if information is a Brownian filtration, if the aggregate consumption rate is an Ito process, and if agents have state independent time-additive utility functions, then equilibrium price processes are Ito processes. In that context, consumption at adjacent dates are perfect nonsubstitutes; cf. Huang and Kreps [12]. The continuity of shadow prices is

arrived at by the assumption on the aggregate endowments.

The question dealt with in this paper is a little different from the works mentioned above. We allow agents to consume at lumps if they wish to. We are not concerned with the existence of an equilibrium per se. Rather, we are interested in characterizing an equilibrium in our economy if indeed one exists. Thus, our work is more in line with Harrison and Kreps [8] and Huang [10]. We get the Ito process representation of price processes from continuity of information and preferences _independently_ of the behavior of aggregate endowments.

It is not difficult either to see that any Arrow-Debreu equilibrium in our economy can be implemented in the sense of Duffie and Huang [5] using a construction similar to Duffie [4]. But securities chosen can pay almost arbitrary patterns of dividends.

We haven't been able to prove the existence of an equilibrium in general in our economy with heterogenous agents. With homogeneous agents, this is not a problem as long as agents' preferences are defined on the whole commodity space. If homogenous agents' preferences are only defined in the positive orthant of V, then there exists an equilibrium as long as the preferences are _proper_. If preferences are τ-proper, the shadow prices are in X. If they are τ^*-proper, the shadow prices are in H. For the definition of properness, see Mas-Colell [16]. For related issues on viable price systems in economics with an infinite dimensional commodity space, see Kreps [15].

7. Concluding Remarks

We have developed a theory of continuous trading for a very general commodity space including all the previously treated models in the literature as special cases. A martingale characterization of an equilibrium price system similar to that of Harrison and Kreps [8] is derived. Sample paths

properties of a price system are linked to agents' preferences and the way in which information is revealed. Under some conditions, the Ito process representation of an equilibrium price process is an inherent property of the Brownian filtration.

The existence of an equilibrium in our economy is still an open question. The issue is that we do not know whether there exists an Arrow-Debreu equilibrium in our economy. If we did, we could demonstrate the existence of an equilibrium by showing that the Arrow-Debreu equilibrium can be implemented by continuous trading of few long-lived securities; cf. Duffie and Huang [5]. Given the work of Mas-Colell [17] the issue seems to be resolvable in the near future.

Appendix I

PROPOSITION A1: $\psi: X x V \to R$,

$$\psi(x,v) = E(\int_{[0,T]} x(t)dv(t))$$

separates points.

PROOF: Let x_1, $x_2 \in X$ and $x_1 \neq x_2$. Since x_1 and x_2 are RCLL processes, we know that, without loss of generality, there must exist a $t\varepsilon[0,T]$ and $A\varepsilon \mathcal{F}_t$ with $P(A) > 0$ such that

$$x_1(\omega,t) > x_2(\omega,t) \qquad \forall\ \omega\varepsilon A \quad .$$

Now define $v: \Omega\ X[0,T] \to R$ by

$$\begin{aligned} v(\omega,s) &= 1 \qquad \forall\ \omega\varepsilon A \quad s \geq t \\ &= 0 \qquad \text{otherwise} . \end{aligned}$$

Then

$$\begin{aligned} \psi(x_1,v) - \psi(x_2,v) &= E(\int_{[0,T]} (x_1(s) - x_2(s))dv(s) \\ &= E((x_1(t) - x_2(t))1_A) > 0 \quad . \end{aligned}$$

Thus V separates points in X.

Next let $v\varepsilon V$, $v\neq 0$. Then there must exist $t\varepsilon[0,T]$ and $A_1 \varepsilon \mathcal{F}_t$ with $P(A_1) > 0$ such that $v(\omega,t) \neq 0 \ \ \forall\ \omega\varepsilon A_1$. Without loss of generality we assume that $v(t) - v(t-) > 0$ on A_1. Now define an optional random variable $\tau_\varepsilon: \Omega \to R_+ U\{+\infty\}$ by

$$\tau_\varepsilon(\omega) = \inf\{s: t \leq s \leq T,\ \varepsilon \leq |v(s) - v(t)| < v(t) - v(t-)\} ,$$

for some $\varepsilon > 0$. We take cases.

CASE 1: There does not exist $\varepsilon > 0$ such that $\tau_\varepsilon \neq +\infty$ on $A_2 \subset A_1$

with $P(A_2) > 0$. Then $v(s) = v(t)$ a.s. $\forall\ s\varepsilon[t,T]$. We define $x\colon \Omega \times [0,T]$ by

$$x(\omega,s) = 1 \qquad \omega\varepsilon A_1 \quad s\varepsilon[t,T]$$
$$= 0 \qquad \text{elsewhere} .$$

Then $x\varepsilon X$ and

$$\psi(x,v) = E(1_{A_1}(v(t) - v(t_1))) > 0 .$$

CASE 2: Suppose that $|v(s) - v(t)| \geq v(t) - v(t-)$ $\forall\ s\varepsilon[t,T]$ a.s. This is impossible by the right continuity of v.

CASE 3: Let $\varepsilon>0$ be such that $\tau_\varepsilon < +\infty$ on $A_3 \subset A_1$ with $P(A_3) > 0$. It is clear that $\tau_\varepsilon > t$. We define $x\colon \Omega \times [0,T] \to R$ by

$$x(\omega,t) = 1_{A_1} \cdot 1_{[t,\tau_\varepsilon(\omega))}(\omega,t) .$$

Since $[t,\tau_\varepsilon)$ is a stochastic interval, x is adapted and, by construction, right continuous with left limits (cf. Chung and Williams [1], Chapter 2). That is, $x\varepsilon X$. Then

$$\psi(x,v) = E(\int_{[0,T]} x(s)dv(s))$$
$$= E(1_{A_1}(v(\tau_\varepsilon) - v(t-))) > 0 .$$

Thus X separates points in V. Q.E.D.

Footnotes

[1]See Dellacherie and Meyer [2], Chapter VII.

[2]See Dellacherie and Meyer [2], Chapter VIII.

REFERENCES

1. K. CHUNG AND R. WILLIAMS, "Introduction to Stochastic Integration," Birkhauser Boston Inc., 1983

2. C. DELLACHERIE AND P. MEYER, "Probabilities and Potential B: Theory of Martingales," North-Holland Publishing Company, New York, New York, 1982

3. C. DELLACHERIE, Un survol de la theorie de l'integral stochastique, Stochastic Processes Appl. 11 (1981), 215-260

4. D. DUFFIE, Existence and spanning number of dynamic equilibria under uncertainty, Stanford University (mimeo), 1984

5. D. DUFFIE AND C. HUANG, Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities, forthcoming in Econometrica, 1984

6. I. GIHMAN AND A. SKOROHOD, "Controlled Stochastic Processes," Springer-Verlag: New York, 1979

7. J.M. HARRISON, Stochastic Calculus and Its Appliations, Lecture Notes, Graduate School of Business, Stanford University, 1982

8. _____ AND D. KREPS, Martingales and arbitrage in multiperiod securities markets, J. Econ. Theor. 20 (1979), 381-408.

9. _____ AND S. PLISKA, Martingales and stochastic integrals in the theory of continuous trading, Stochastic Processes Appl. 11 (1981), 215-260

10. C. HUANG, Information structure and equilibrium asset prices, forthcoming in J. Econ. Theor., 1984

11. _____, An intertemporal general equilibrium asset pricing model: the case of diffusion information, MIT (mimeo), 1983

12. _____ AND D. KREPS, Intertemporal preferences with a continuous time dimension: An exploratory study, MIT (mimeo), 1983

13 D. KREPS, Multiperiod securities and the efficient allocation of risk: a comment on the Black-Scholes option pricing model in J. McCall, "The Economics of Uncertainty and Information," University of Chicago Press, 1982

14. _____, Three essays on capital markets, Technical Report 298, Institute for Mathematical Studies in The Social Sciences, Stanford University, 1979

15. _____, Arbitrage and equilibrium in economies with infinitely many commodities, J. Math. Econ. 8 (1981), 15-35

16. A. MAS-COLELL, The price equilibrium existence problem in Banach lattices, Harvard University (mimeo), 1983

17. _______, Notes on Pareto optima in linear spaces, Harvard University (mimeo), 1984

18. P. MEYER, Un cours sur les integrales stochastiques, in "Seminaires de Probabilite X, Lecture Notes in Mathematics 511," Springer-Verlag, New York, 1976

19. R. RADNER, Existence of equilibrium of plans, prices and price expectations in a sequence of markets, Econometrica 40 (1972), 289-303

20. H. ROYDEN, "Real Analysis," 2nd ed., Macmillan Publishing Co., Inc., New York, 1968

21. H. SCHAEFER, "Topological Vector Spaces," Macmillan, New York, 1966